

Group Decision Making with Incomplete Reciprocal Preference Relations Based on Multiplicative Consistency *

Etienne E. Kerre^{1 †}, Atiq-ur-Rehman², Samina Ashraf³

¹ *Fuzziness and Uncertainty Modeling Research Unit,
B-9000 Gent, Belgium[‡]*

E-mail: etienne.kerre@ugent.be

² *COMSATS Institute of Information Technology,
Department of Mathematics,
Lahore, Punjab, Pakistan*

E-mail: atiq.math@gmail.com

³ *Director, Quality Enhancement cell,
Lahore College for Women University
Lahore, Punjab, Pakistan*

E-mail: saminaa561@gmail.com

Abstract

This paper comprises a new iterative method for multi-person decision making based on multiplicative consistency with incomplete reciprocal preference relations (IRPRs). Additionally, multiplicative transitivity property of reciprocal preference relation (RPR) is used at the first level to estimate the unknown preference values and get the complete preference relation, then it is confirmed to be multiplicative consistent by using transitive closure formula. Following this, expert's weights are evaluated by merging consistency and trust weights. The consistency weights against the experts are evaluated through multiplicative consistency investigation of the preferences given by each expert, while trust weights play the role to measure the level of trust for an expert. The consensus process determines whether the selection procedure should start or not. If it results in negative, the feedback mechanism is used to enhance the consensus degree. At the end, a numerical example is given to demonstrate the efficiency and practicality of the proposed method.

Keywords: Fuzzy set; Group decision making; Reciprocal preference relation; Incomplete preference relation; Multiplicative consistency.

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[†] Etienne E. Kerre, Fuzziness and Uncertainty Modeling Research Unit B-9000 Gent, Belgium.

[‡] etienne.kerre@ugent.be

1. Introduction

Decision making (DM) is an intellectual procedure which is used to select the best option(s) amongst several different options, it initiates when we have to do something but do not know what. Every individual faces DM situations in his/her daily life: common examples for these situations are shopping, to choose what to eat, and deciding whom or what to vote for in an election or referendum, and can be categorized in several different groups according to certain characteristics as the source(s) for the information and the preference representation formats that are used to solve the decision problem. In our framework, the selection of the best alternative(s) from a predetermined set $X = \{x_1, x_2, x_3, \dots, x_n\}, n \geq 2$ of possible alternatives is the goal.

DM is not only the case for a single expert, where he/she compares a finite set of alternatives and construct a preference relation, but some decision problems have to be solved by a group of experts who work together to find the best alternative(s) from a set of feasible alternatives. This decision making with multiple experts is called group decision making (GDM) or also known as multi-person decision making. To solve a GDM problem appropriately, two key processes play an essential role:

- (i) consensus process;
- (ii) selection process.

The prior is an iterative process which is composed of several consensus rounds, where the experts accept to negotiate diverse sentiments to have an acceptable level, but a unanimous or full consensus is often not achievable in practice¹³. After getting the experts' opinions close enough, the selection process takes place which aims to rank and select (a) suitable alternative(s) from a given set of feasible alternatives. When a number of experts interact to reach a decision, each expert may have exclusive inspirations or objectives and a different decision procedure but has a common interest in approaching to select the "best" option(s).

In modern era, various consensus models have been proposed in literature for GDM against a number of preference relations^{2,9,12,25}. However, there

may arise some situations for DM problems in which experts are unable to provide precise and complete assessments due to the pressure to make a quick decision, Complexity, or incomprehensive information against the problem to be explained, such situations result in incomplete fuzzy preference relations (IFPRs) i.e., some preference values are missing. GDM in IFPRs environment has been receiving an intensive interest of researchers, and various procedures have been presented to determine unknown preference values¹⁷. Such as, a least squared procedure was proposed by Gong⁸ in 2008 to determine the priority vector for GDM in incomplete preference relations' (IPRs) environment. In 2010, a goal programming model was presented by Fan and Zhang⁷ for GDM to deal with IPRs in three formats. In 2013, Xu et al. proposed logarithmic least squares method to evaluate the priority weights in GDM dealing with IFPRs and develop the acceptable fuzzy consistency ratio²⁰. Xu and Wang²¹ in 2013, presented eigenvector method to repair an IFPR with consistency test relation. In 2015, Xu et al.²² presented a least deviation method to evaluate the priority weights for GDM in IRPRs environment. In 2015, the trust based consensus model and aggregation method for GDM were investigated by Wu et al.¹⁹ in the context of incomplete linguistic information.

Moreover, consistency is an important issue to accept when data are provided by the experts^{5,10} and is linked to the transitivity property. In 2007, Herrera-Viedma et al.¹² presented an additive consistency based iterative scheme to evaluate the missing preferences in IFPRs. In 2008, Alonso et al.¹ extended the idea proposed by Herrera-Viedma et al.¹² to investigate the missing information against several preference formats. In 2014, a consensus based model was proposed by Wu and Chiclana¹⁸ for multiplicative consistency of reciprocal intuitionistic preference relations and discussed its application to estimate the unknown preference degrees. In 2015, a confidence consistency driven approach was proposed by Ureña et al.¹⁶ to handle GDM problems with incomplete reciprocal intuitionistic preference relations. This approach deals with confidence level and implemented for both consistency

and confidence in the determination procedure. In 2015, Ashraf et al.³ proposed a new method for GDM with IFPRs based on T -consistency and the order consistency, where T stands for a triangular norm. In 2017, Xu et al.²³ developed a consensus based model for hesitant FPRs and used it in water allocation management as an application.

To achieve an acceptable consensus level, feedback mechanism plays an important role in certain consensus measures^{6,25,26,27}. In²⁵, Zhang et al. developed a consensus building method based on multiplicative consistency for GDM with IRPRs. Inspired by the work of Zhang et al.²⁵, we observed that the “multiplicative transitivity property” i.e., $\frac{r_{ik}}{r_{ki}} = \frac{r_{ij}r_{jk}}{r_{ji}r_{kj}}$ for all $j \neq i, k \in \{1, 2, 3, 4\}$ is violated or overlooked. For example, if we consider the aggregated group preference matrix constructed in²⁵:

$$P = \begin{bmatrix} 0.5 & 0.43 & 0.58 & 0.74 \\ 0.57 & 0.5 & 0.75 & 0.83 \\ 0.42 & 0.25 & 0.5 & 0.74 \\ 0.26 & 0.17 & 0.26 & 0.5 \end{bmatrix} \quad (1)$$

the multiplicative transitivity for, say, $i = 1$ and $k = 2$ implies that $\frac{r_{12}}{r_{21}} = \frac{r_{1j}r_{j2}}{r_{j1}r_{2j}}$ must be satisfied for $j = 3, 4$, but from matrix P we have: $\frac{r_{12}}{r_{21}} = 0.7543859649$ and $\frac{r_{13}r_{32}}{r_{31}r_{23}} = 0.4603174603$, $\frac{r_{14}r_{42}}{r_{41}r_{24}} = 0.5829471733$, we can see that the aggregated matrix constructed by Zhang et al.²⁵ does not satisfy the multiplicative consistency. We believe that the multiplicative transitivity property is very hard to satisfy while decimal numbers are being used but to ensure consistency the error can be minimized upto an insignificant level.

In this paper, we present an improved method for consensus building in group decision making based on multiplicative consistency with IRPRs. At the first step, we evaluate the missing preferences of IRPRs based on the multiplicative transitivity. Then, we construct the modified consistency matrices of experts which satisfy the multiplicative consistency and measure the level of consistency. The degrees of importance are assigned to experts based on consistency weights aggregated with trust weights. The proposed method provides us with a valuable way for consensus building in group decision making based on multiplicative consistency with incomplete preference relations.

The rest of the paper is organized as follows. In Section 2, we focus on some preliminaries used in this paper. In Section 3, we present a new procedure to estimate the missing preferences in incomplete preference relations based on the multiplicative transitivity and construct the modified consistency matrices which satisfy the multiplicative consistency. In Section 4, the proposed GDM process is detailed. In Section 5, an example is given to illustrate the realism and achievability of the proposed technique. Last section includes some conclusions.

2. Preliminaries

In 1965, Zadeh introduced fuzzy set theory²⁴, designated with a number between 0 and 1, to cope with imprecise and uncertain information working in complex situation.

Definition 1. *Fuzzy Set*²⁴: A fuzzy set A on the universe of discourse X is a mapping from X to $[0, 1]$, and is denoted by $A = \{(x, A(x))\}$. For any $x \in X$, the value $A(x)$ is called the degree of membership of x in A i.e., $A(x) = \text{Degree}(x \in A)$, and the map $A : X \rightarrow [0, 1]$ is called a membership function.

Definition 2. *Reciprocal Preference Relation (RPR)*⁵: An RPR R on a set X of alternatives $X = \{x_1, x_2, \dots, x_n\}$ is characterized by a membership function $R(x_i, x_j) = r_{ij}$, satisfying $r_{ij} + r_{ji} = 1$ for all $i, j \in \{1, 2, 3, \dots, n\}$.

An RPR may be conveniently denoted by matrix $R = (r_{ij})_{n \times n}$, with the following interpretations:

$$R = (r_{ij})_{n \times n} = \begin{bmatrix} 0.5 & r_{12} & \cdot & \cdot & r_{1n} \\ r_{21} & 0.5 & \cdot & \cdot & r_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{n1} & r_{n2} & \cdot & \cdot & 0.5 \end{bmatrix} \quad (2)$$

where r_{ij} represents the preference value of alternative i over alternative j , and all $r_{ii} = 0.5$. If $r_{ij} = 0.5$, this indicates that there is no difference between the two alternatives; if $r_{ij} > 0.5$, it implies that alternative i is superior to alternative j .

Definition 3. *Incomplete Reciprocal Preference Relation (IRPR)*¹¹: An IRPR $R = (r_{ij})_{n \times n}$ carries at

least one unknown preference value r_{ij} for which the expert does not have a clear idea of the degree of preference of alternative x_i over the alternative x_j .

Definition 4. *Multiplicative Transitivity Property*¹⁵: An RPR $R = (r_{ij})_{n \times n}$ on a finite set X of alternatives is said to be multiplicative transitive if and only if

$$\frac{r_{ij}}{r_{ji}} = \frac{r_{ik}r_{kj}}{r_{jk}r_{ki}} \quad (3)$$

holds for all $i, j, k \in \{1, 2, 3, \dots, n\}$ with non-zero preference values. The multiplicative transitivity leads to multiplicative consistency of RPR.

3. Evaluating missing preferences

This section puts forward a new scheme to estimate the unknown preference values in an IRPR based on multiplicative transitivity. Moreover, the proposed procedure is used to construct a multiplicative consistent matrix.

So as to determine missing preferences in an IRPR $R = (r_{ij})_{n \times n}$, the pairs of alternatives for known and unknown preferences are signified in form of following sets:

$$K_v = \{(i, j) | r_{ij} \text{ is known}\}; \quad (4)$$

$$U_v = \{(i, j) | r_{ij} \text{ is unknown}\}, \quad (5)$$

where K_v is the set of pairs of alternatives with known preference degrees whereas U_v represents the set of pairs of alternatives with missing preference degrees. The preference value of alternative x_i over x_j belongs to $[0, 1]$ (i.e., $r_{ij} \in [0, 1]$). Since $r_{ij} + r_{ji} = 1$ for $1 \leq i \leq n$ and $1 \leq j \leq n$, therefore, multiplicative transitivity (3) can be written as:

$$r_{ij} = \frac{r_{ik} \cdot r_{kj}}{1 - r_{ik} - r_{kj} + 2r_{ik} \cdot r_{kj}}, \quad (6)$$

where $1 > r_{ij} > 0 \forall i, j \in \{1, 2, 3, \dots, n\}$. Hence, we can define following set of intermediate alternative x_j which can be used to determine the unknown preference value r_{ik} of alternative x_i over alternative x_k :

$$W_{ij} = \{k \neq i, j \mid (i, k) \in K_v, (k, j) \in K_v \text{ and } (i, j) \in U_v\}, \quad (7)$$

for $1 \leq i \leq n$, $1 \leq j \leq n$ and $1 \leq k \leq n$. As consistency is an essential property for RPR, expression (6) can be used to estimate the missing preference value by using other preference values to maintain the internal consistency of RPR. Definitely, the unknown preference value r_{ij} ($i \neq j$) can be calculated using an intermediate alternative x_k ($k \neq i, j$) based on (6), but the aggregated value (global value) of r_{ij} is obtained by using the max aggregation operator and is the degree of preference of alternative x_i over the alternative x_j :

$$r_{ij} = \begin{cases} \max_{k \in W_{ij}} \left(\frac{r_{ik} \cdot r_{kj}}{1 - r_{ik} - r_{kj} + 2r_{ik} \cdot r_{kj}} \right), & \text{if } |W_{ij}| \neq 0 \\ 0.5, & \text{otherwise} \end{cases} \quad (8)$$

where $|W_{ij}|$ is the cardinality of the set W_{ij} . Aggregation and fusion of information are basic concerns for all kinds of knowledge based systems, from image processing to decision making, from pattern recognition to machine learning. From a general point of view we can say that aggregation has for purpose the simultaneous use of different pieces of information (provided by several sources) in order to come to a conclusion or a decision. Here max aggregation operator is used for optimal attitude while other operators as min and *average* etc can also be used. The value r_{ji} is estimated by using reciprocity after having the value of r_{ij} as:

$$r_{ji} = 1 - r_{ij} \quad (9)$$

New sets of the pairs of alternatives for known and unknown preference values are evaluated as follows:

$$K'_v = K_v \cup \{(i, j)\}, \quad (10)$$

$$U'_v = U_v - \{(i, j)\}. \quad (11)$$

After having a complete RPR, it needs to be fully multiplicative consistent RPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ which can be obtained by calculating \tilde{r}_{ij} as:

$$\tilde{r}_{ij} = \max_{k \neq i, j} \left(r_{ij}, \frac{r_{ik} \cdot r_{kj}}{1 - r_{ik} - r_{kj} + 2r_{ik} \cdot r_{kj}} \right) \quad (12)$$

such that \tilde{R} is stable with $\tilde{r}_{ij} + \tilde{r}_{ji} = 1$. It remains in work until two consecutive outcomes result in same RPRs regarding their preferences, at this stage relation \tilde{R} will satisfy the multiplicative consistency property.

4. Iterative procedure for GDM

Now we turn towards our major task to develop a GDM procedure in IRPRs environment. In this section, a new step-by-step algorithm is presented for GDM based on multiplicative consistency. An explanatory example is given to validate the technique. For ease, the structure of the resolution process is also shown in Figure 1.

Suppose that there are n alternatives x_1, x_2, \dots, x_n and m experts E_1, E_2, \dots, E_m . Let R^q be the fuzzy preference relation for the expert E_q shown as follows:

$$R^q = (r_{ij}^q)_{n \times n} = \begin{bmatrix} 0.5 & r_{12}^q & \cdot & \cdot & r_{1n}^q \\ r_{21}^q & 0.5 & \cdot & \cdot & r_{2n}^q \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{n1}^q & r_{n2}^q & \cdot & \cdot & 0.5 \end{bmatrix},$$

where $r_{ij}^q \in [0, 1]$ is the preference value given by expert E_q for alternative x_i over x_j , $r_{ij}^q + r_{ji}^q = 1$, $1 \leq i \leq n$, $1 \leq j \leq n$ and $1 \leq q \leq m$ and $r_{ii}^q = 0.5$, for all $i \in \{1, 2, \dots, n\}$ as an alternative cannot be preferred on itself. Some of the preference matrices given by the experts may be IRPRs, because of time constraint or lack of information and complexity. The proposed GDM technique consists of several stages which are described as follows:

4.1. Estimating missing preferences

To determine the missing preference values of an IRPR R^q given by the expert E_q , initially, the sets K^q and U^q of pairs of alternatives for known and unknown preferences are introduced as in (4) and (5) respectively. After this, the multiplicative transitivity based preference values are estimated by using (7)-(11) to construct the complete RPR R^q .

4.2. Consistency measures

After evaluating the complete RPRs, their parallel multiplicative consistent RPRs \tilde{R}^q , for $q = 1, 2, 3, \dots, m$, can also be obtained with the help of (12). We can then approximate the degree of consistency of an RPR R^q based on its similarity with

the corresponding multiplicative transitivity based \tilde{R}^q by computing their distances²⁵.

1. Multiplicative consistency index (MCI) of a pair of alternatives is determined by using:

$$MCI(r_{ij}^q) = 1 - d(r_{ij}^q, \tilde{r}_{ij}^q) \quad (13)$$

where $d(r_{ij}^q, \tilde{r}_{ij}^q)$ is the error (distance) measured by $\varepsilon r_{ij}^q = d(r_{ij}^q, \tilde{r}_{ij}^q) = |r_{ij}^q - \tilde{r}_{ij}^q|$. When $MCI(r_{ij}^q) = 1$ this implies that $\varepsilon r_{ij}^q = 0$ and there is no inconsistency at all. Apparently, the higher the value of $MCI(r_{ij}^q)$, the more consistent r_{ij}^q is with respect to the rest of the preference values involving alternatives x_i and x_j .

2. MCI associated to a particular alternative x_i , $1 \leq i \leq n$, of an RPR is evaluated by:

$$MCI(x_i) = \frac{1}{2(n-1)} \sum_{j=1}^n (MCI(r_{ij}^q) + MCI(r_{ji}^q)) \quad (14)$$

with $MCI(x_i) \in [0, 1]$. When $MCI(x_i) = 1$ all the preferences involving the alternative x_i are fully consistent, otherwise, the lower $MCI(x_i)$ the more inconsistent these preference values are.

3. Finally we determine MCI of an RPR R^q by taking the average of all MCI of alternatives x_i :

$$MCI(R^q) = \frac{1}{n} \sum_{i=1}^n MCI(x_i) \quad (15)$$

with $MCI(R^q) \in [0, 1]$. When $MCI(R^q) = 1$ the preference relation R^q is fully consistent, otherwise, the lower $MCI(R^q)$ the more inconsistent R^q is.

As soon as the MCI is computed in three levels using expressions (13)-(15), it is rational to assign the higher weights to the experts against the preference relations with larger consistency degrees respectively. Hence, consistency weights can be assigned to the experts by using the relation:

$$Cw(E_q) = \frac{MCI(R^q)}{\sum_{q=1}^m MCI(R^q)} \quad (16)$$

where m represents the number of experts, and $MCI(R^q)$ is the MCI of the opinion of E_q .

4.3. Allocating weights to experts

The trust weight $tw(E_q)$ is the degree of trust on each expert by others such that $\sum_{q=1}^m tw(E_q) = m$. Convincingly, the experts with larger trust weights can have higher weight in the aggregation process. Primarily, the same trust weight $tw(E_q)$ has been assigned to each expert. But the trust weights of experts do not remain the same in each consensus round, the procedure given by Zhang et al.²⁵ will be used to evaluate the trust weights. Final weights to experts are allocated by emerging respective trust weights and consistency weights under the use of following relation²⁵:

$$w(E_q) = \frac{tw(E_q) \times Cw(E_q)}{\sum_{q=1}^m tw(E_q) \times Cw(E_q)} \quad (17)$$

with $\sum_{q=1}^m w(E_q) = 1$.

4.4. Consensus measures

After having the RPRs with complete information, it is necessary to measure the consensus among the experts. Regarding this, similarity matrices $S^{qr} = (s_{ij}^{qr})_{n \times n}$ for every pair of experts (E_q, E_r) ($q = 1, 2, \dots, m-1; r = q+1, \dots, m$) are to be determined and defined as:

$$s_{ij}^{qr} = 1 - d(r_{ij}^q, r_{ij}^r) \quad (18)$$

where $d(r_{ij}^q, r_{ij}^r) = |r_{ij}^q - r_{ij}^r|$. Then the collective similarity matrix $S = (s_{ij})_{n \times n}$ is constructed after aggregating all the similarity matrices by using following relation.

$$s_{ij} = \frac{2}{m(m-1)} \sum_{q=1}^{m-1} \sum_{r=q+1}^m s_{ij}^{qr} \quad (19)$$

Three different levels are involved to compute the degree of consensus amongst the experts as follows²⁵:

1. At first level, the consensus degree on a pair of alternative (x_i, x_j) , denoted by cod_{ij} is defined to estimate the degree of consensus amongst all experts on that pair of alternatives:

$$cod_{ij} = s_{ij} \quad (20)$$

2. At second level, the consensus degree on alternative x_i , denoted by CoD_i is defined to determine the consensus degree amongst all the experts on that alternative:

$$CoD_i = \frac{1}{2(n-1)} \sum_{j=1, j \neq i}^n (s_{ij} + s_{ji}) \quad (21)$$

3. At third level, the consensus degree on the relation, denoted by CoR is defined to calculate the global degree of consensus amongst all the experts judgments:

$$CoR = \frac{1}{n} \sum_{i=1}^n CoD_i \quad (22)$$

Once the global consensus level among all the experts is reached, it requires to compare with a threshold consensus degree η , generally settled in advance depending upon the nature of problem. If $CoR \geq \eta$, this shows that an acceptable level of consensus has been obtained, and the decision process begins. Otherwise, the consensus degree is not stable, and feedback mechanism is originated.

4.5. Feedback mechanism

The central aim of feedback mechanism is to provide comprehensive knowledge to experts, so as to change their opinions acceptably to enhance the consensus degree. When consensus is not sufficiently high, then we have to identify the preference values that are to be changed, and following formula helps us in this regard:

$$R^q = \{(i, j) \mid cod_{ij} < CoR \text{ and } r_{ij}^q \text{ is known}\} \quad (23)$$

for $i, j \in \{1, \dots, n\}$. The system recommends that the corresponding expert has to increase value if it is smaller than the mean value of the valuations of the

rest of experts, or decrease it if it is greater than the mean²⁵. All at once, there may be a possibility that the minority opinion is precise⁴, a conflict resolution method has been proposed to handle such a situation which contains correct opinions against few experts¹⁴. We need to find out all alternatives x_i whose consensus measures CoD_i are lower than CoR , and then elaborate the particular pair of experts whose opinions contradict one another on alternative x_i to the greatest extent, (i.e., with the largest deviation of S on alternatives x_i , which can be measured by the sum of $\sum_{j=1, j \neq i}^n (s_{ij} + s_{ji})$). With this, we can make them change their initial preferences and, at the same time, accelerate the consensus process.

4.6. Accumulation phase

It may quite frequently happen that the preference degree set forth by each expert is weighted differently. As soon as the weights for the experts are estimated, their preferences need to be accumulated into a global one. Determine the collective matrix R^C against all experts, shown as follows:

$$R^C = (r_{ij}^c)_{n \times n} = \left(\sum_{q=1}^m w(E_q) \times \tilde{r}_{ij}^q \right)_{n \times n} \quad (24)$$

where $1 \leq i \leq n, 1 \leq j \leq n$.

4.7. Selection phase

After reaching a satisfactory consensus degree amongst all experts, the selection process is initiated to rank all the alternatives in order to select the best option. For a consistent RPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$, the ranking value $Rv(x_i)$ of alternative x_i , $i = 1, \dots, n$, is defined by:

$$Rv(x_i) = \frac{2}{n(n-1)} \sum_{j=1, j \neq i}^n \tilde{r}_{ij}, \quad i = 1, \dots, n \quad (25)$$

with $\sum_{i=1}^n Rv(x_i) = 1$.

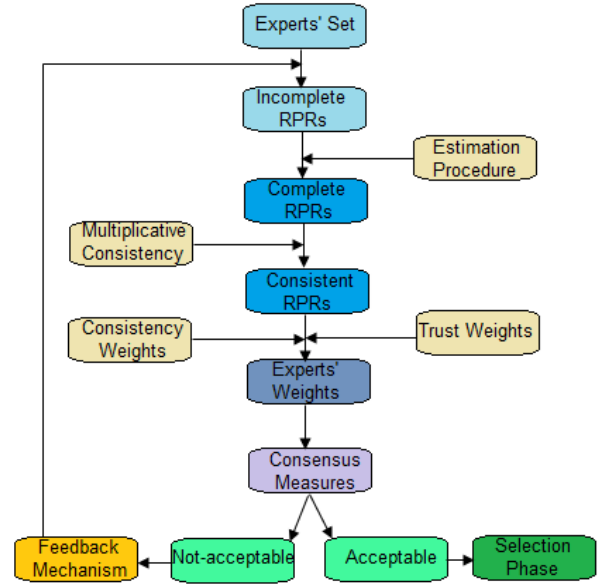


Fig. 1. Resolution process for GDM with IFPRs

5. Numerical example

This section deals with a numerical example taken from²⁵ in order to demonstrate the process of the proposed method and its effectiveness.

Consider that four experts E_1, E_2, E_3 and E_4 from different fields are requested to select the best alternative out of four alternatives x_1, x_2, x_3, x_4 . The four experts give their RPRs as follows:

$$R^1 = \begin{bmatrix} 0.5 & 0.6 & r_{13}^1 & r_{14}^1 \\ 0.4 & 0.5 & 0.7 & r_{24}^1 \\ r_{31}^1 & 0.3 & 0.5 & 0.9 \\ r_{41}^1 & r_{42}^1 & 0.1 & 0.5 \end{bmatrix}, \quad R^2 = \begin{bmatrix} 0.5 & 0.6 & 0.7 & r_{14}^2 \\ 0.4 & 0.5 & r_{23}^2 & 0.7 \\ 0.3 & r_{32}^2 & 0.5 & r_{34}^2 \\ r_{41}^2 & 0.3 & r_{43}^2 & 0.5 \end{bmatrix},$$

$$R^3 = \begin{bmatrix} 0.5 & 0.3 & 0.5 & 0.75 \\ 0.7 & 0.5 & 0.8 & 0.6 \\ 0.5 & 0.2 & 0.5 & 0.8 \\ 0.25 & 0.4 & 0.2 & 0.5 \end{bmatrix}, \quad R^4 = \begin{bmatrix} 0.5 & 0.4 & 0.55 & 0.65 \\ 0.6 & 0.5 & 0.8 & 0.75 \\ 0.45 & 0.2 & 0.5 & 0.7 \\ 0.35 & 0.25 & 0.3 & 0.5 \end{bmatrix}.$$

The threshold consensus level η settled in advance is 0.80. Now, we perform the following steps to evaluate the result:

Step-i: Estimating the missing preferences

Initially, all the missing preference values need to be determined using the multiplicative transitivity property mentioned in Section 1.

Taking R^1 , for example. The sets of pairs of alternatives for known and unknown preference values are determined as follows:

$$\begin{aligned} K_v^1 &= \{(1,2), (2,1), (2,3), (3,2), (3,4), (4,3)\}, \\ U_v^1 &= \{(1,3), (3,1), (1,4), (4,1), (2,4), (4,2)\}. \end{aligned}$$

All the missing preference values are calculated under the use of (7)-(11) to complete the given IRPR. Hence, the complete RPR R^1 against expert E_1 is obtained as follows:

$$R^1 = \begin{bmatrix} 0.5000 & 0.6000 & 0.7778 & 0.9692 \\ 0.4000 & 0.5000 & 0.7000 & 0.9545 \\ 0.2222 & 0.3000 & 0.5000 & 0.9000 \\ 0.0308 & 0.0455 & 0.1000 & 0.5000 \end{bmatrix}.$$

Similarly, the complete form of R^2 can be obtained and given as:

$$R^2 = \begin{bmatrix} 0.5000 & 0.6000 & 0.7000 & 0.7778 \\ 0.4000 & 0.5000 & 0.6987 & 0.7000 \\ 0.3000 & 0.3913 & 0.5000 & 0.6000 \\ 0.2222 & 0.3000 & 0.4000 & 0.5000 \end{bmatrix}.$$

Step-ii: Consistency analysis

Consistency analysis is being conducted to allocate consistency weights to the experts. For this purpose, all complete RPRs are to be converted into their multiplicative consistent forms by using (12), and are given below:

$$\begin{aligned} \tilde{R}^1 &= \begin{bmatrix} 0.5000 & 0.6000 & 0.7777 & 0.9692 \\ 0.4000 & 0.5000 & 0.6999 & 0.9545 \\ 0.2223 & 0.3001 & 0.5000 & 0.9000 \\ 0.0308 & 0.0455 & 0.1000 & 0.5000 \end{bmatrix}, \\ \tilde{R}^2 &= \begin{bmatrix} 0.5000 & 0.6000 & 0.7000 & 0.7778 \\ 0.4000 & 0.5000 & 0.6987 & 0.7000 \\ 0.3000 & 0.3913 & 0.5000 & 0.6000 \\ 0.2222 & 0.3000 & 0.4000 & 0.5000 \end{bmatrix}, \\ \tilde{R}^3 &= \begin{bmatrix} 0.5000 & 0.4304 & 0.5994 & 0.7916 \\ 0.5696 & 0.5000 & 0.6645 & 0.8341 \\ 0.4006 & 0.3355 & 0.5000 & 0.7174 \\ 0.2084 & 0.1659 & 0.2826 & 0.5000 \end{bmatrix}, \\ \tilde{R}^4 &= \begin{bmatrix} 0.5000 & 0.3504 & 0.5892 & 0.7375 \\ 0.6496 & 0.5000 & 0.7267 & 0.8389 \\ 0.4108 & 0.2733 & 0.5000 & 0.6620 \\ 0.2625 & 0.1611 & 0.3380 & 0.5000 \end{bmatrix}. \end{aligned}$$

The significant MCI values of the experts are evaluated using (13)-(15), as:

$$\begin{aligned} MCI(R^1) &= 0.9999, MCI(R^2) = 1, \\ MCI(R^3) &= 0.8794, MCI(R^4) = 0.9372. \end{aligned}$$

Finally, the consistency weights to the experts are computed by using (16), as:

$$\begin{aligned} Cw(E_1) &= 0.2620, Cw(E_2) = 0.2620, \\ Cw(E_3) &= 0.2304, Cw(E_4) = 0.2456. \end{aligned}$$

Step-iii: Weights to experts

Primarily, all experts are assigned the same trust weights: $tw(E_1) = 1$, $tw(E_2) = 1$, $tw(E_3) = 1$ and $tw(E_4) = 1$. Therefore, the weights of the experts remain the same in the first round as the consistency weights based on (17), as: :

$$\begin{aligned} w(E_1) &= 0.2620, w(E_2) = 0.2620, \\ w(E_3) &= 0.2304, w(E_4) = 0.2456. \end{aligned}$$

Step-iv: Consensus measures

After getting complete RPRs, a mutual similarity relation is computed by aggregating the different similarity matrices among the experts using (18)-(19). Then, the consensus measures are computed at the three levels using (20)-(22).

$$\begin{aligned} S^{12} &= \begin{bmatrix} 1.0 & 1.0 & 0.9223 & 0.8086 \\ 1.0 & 1.0 & 0.9088 & 0.7455 \\ 0.9223 & 0.9088 & 1.0 & 0.7000 \\ 0.8086 & 0.7455 & 0.7000 & 1.0 \end{bmatrix}, \\ S^{13} &= \begin{bmatrix} 1.0 & 0.8304 & 0.8217 & 0.8224 \\ 0.8304 & 1.0 & 0.9646 & 0.8796 \\ 0.8217 & 0.9646 & 1.0 & 0.8174 \\ 0.8224 & 0.8796 & 0.8174 & 1.0 \end{bmatrix}, \\ S^{14} &= \begin{bmatrix} 1.0 & 0.7504 & 0.8115 & 0.7683 \\ 0.7504 & 1.0 & 0.9732 & 0.8844 \\ 0.8115 & 0.9732 & 1.0 & 0.7620 \\ 0.7683 & 0.8844 & 0.7620 & 1.0 \end{bmatrix}, \\ S^{23} &= \begin{bmatrix} 1.0 & 0.8304 & 0.8994 & 0.9862 \\ 0.8304 & 1.0 & 0.9442 & 0.8659 \\ 0.8994 & 0.9442 & 1.0 & 0.8826 \\ 0.9862 & 0.8659 & 0.8826 & 1.0 \end{bmatrix}, \\ S^{24} &= \begin{bmatrix} 1.0 & 0.7504 & 0.9892 & 0.9597 \\ 0.7504 & 1.0 & 0.8820 & 0.8611 \\ 0.9892 & 0.8820 & 1.0 & 0.9380 \\ 0.9597 & 0.8611 & 0.9380 & 1.0 \end{bmatrix}, \end{aligned}$$

$$S^{34} = \begin{bmatrix} 1.0 & 0.9200 & 0.9898 & 0.9459 \\ 0.9200 & 1.0 & 0.9378 & 0.9952 \\ 0.9898 & 0.9378 & 1.0 & 0.9446 \\ 0.9459 & 0.9952 & 0.9446 & 1.0 \end{bmatrix}.$$

1. On pair of alternatives:

$$CoD = \begin{bmatrix} 1.0 & 0.8469 & 0.8890 & 0.8819 \\ 0.8469 & 1.0 & 0.9351 & 0.8719 \\ 0.8890 & 0.9351 & 1.0 & 0.8408 \\ 0.8819 & 0.8719 & 0.8408 & 1.0 \end{bmatrix}$$

2. On alternatives:

$$CoD_1 = 0.8726, CoD_2 = 0.8846,$$

$$CoD_3 = 0.8883, CoD_4 = 0.8649.$$

3. On relation:

$$CoR = 0.8776.$$

Now, the threshold consensus degree η settled in advance is compared with global consensus degree CoR of the relation; $CoR > \eta$. This indicates that the given consensus degree is acceptable amongst the experts, and we have to enter into accumulation phase.

Step-v: Accumulation phase

$$R^C = \begin{bmatrix} 0.5000 & 0.4992 & 0.6690 & 0.8370 \\ 0.5008 & 0.5000 & 0.6697 & 0.8374 \\ 0.3310 & 0.3303 & 0.5000 & 0.7176 \\ 0.1630 & 0.1626 & 0.2824 & 0.5000 \end{bmatrix} \quad (26)$$

Step-vi: Selection phase

The relation R^C obtained in (26) is clearly multiplicative consistent, therefore, apply (25) to estimate the ranking value $R_v(x_i)$ of alternative x_i , $1 \leq i \leq 4$ as follows:

$$R_v(x_1) = 0.3342, R_v(x_2) = 0.3347,$$

$$R_v(x_3) = 0.2298, R_v(x_4) = 0.1013,$$

where $\sum_{i=1}^4 R_v(x_i) = 1$ and $R_v(x_2) > R_v(x_1) > R_v(x_3) > R_v(x_4)$, therefore, the ranking order of alternatives x_1, x_2, x_3 and x_4 is: $x_2 > x_1 > x_3 > x_4$, and the best option is x_2 .

To validate the effectiveness of our proposed method, we compare the results with Zhang et al.²⁵ model which yielded the same results i.e., $x_2 > x_1 > x_3 > x_4$ as ours. But the aggregated matrix obtained by our method is multiplicative consistent at least correct to three decimal places which can further be extended i.e., if we take $i = 1$ and $k = 2$, the preference value in aggregated matrix R^C is $r_{12} = 0.4992$ and the multiplicative consistency requires the same or closed output to r_{12} by $\frac{r_{1j}r_{j2}}{r_{j1}r_{2j}}$ for $j = 3, 4$ as:

$$\frac{r_{13}r_{32}}{r_{31}r_{23}}r_{21} = 0.4992,$$

$$\frac{r_{14}r_{42}}{r_{41}r_{24}}r_{21} = 0.4993.$$

which shows that $r_{12} \approx \frac{r_{1j}r_{j2}}{r_{j1}r_{2j}}r_{21}$. Therefore, this check can be made for the whole matrix.

6. Conclusion

In this paper, an improved hybrid consensus method for GDM problems based on incomplete RPRs is proposed. The multiplicative transitive property is used to estimate the missing values and transitive closure formula is used to make the matrices multiplicative consistent. The weights of the experts are obtained from the consistency analysis and a calculation of degree of trust. Rationally, the experts with high level of consistency and substantial trust degree should have to assign large weights, in order carry more importance in the aggregation process. Additionally, a feedback mechanism making advice to experts subject to their trust weights and consistency weights was proposed which can accelerate the execution of a higher consensus degree. After reaching a satisfactory consensus degree amongst all experts, the selection process is initiated to rank all the alternatives in order to select the best option. Some numerical examples were provided to highlight the efficiency and feasibility of the proposed method, and some results in comparison with model proposed by Zhang et al.²⁵ were given. The results established the practicability of the method, which can help us to gain a greater insight into the GDM process.

To summarize, some of the major advantages of the proposed technique are as follows:

(1). In the proposed method, the missing preference values for RPRs were estimated using multiplicative transitivity which is more suitable to attain consistency of RPRs as compared with other consistency-based methods; (2). The trustworthiness of experts weights were improved by combining consistency weights and trust weights, and used to measure the consistency of experts estimations upto acceptable level; (3). The proposed method resulted in highly consistent preference relations as compare to other models²⁵.

To the best of our knowledge, there are only few hybrid methods of this kind which have been proposed in literature to deal with GDM problems in incomplete RPRs environment. We think that this method handles GDM problems more efficiently and yields more effective agreements.

In future, we will extend the proposed model for multi-criteria decision making.

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